

# A GEOMETRICALLY-EXACT ROD MODEL INCORPORATING SHEAR AND TORSION-WARPING DEFORMATION

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**Abstract**—A fully nonlinear, three-dimensional rod model is developed that incorporates transverse shear and torsion-warping deformation. The geometric setting is that of a constrained body model with configuration space modeled on  $\mathbb{R}^3 \times SO(3) \times \mathbb{R}$ ; a differentiable manifold. The proposed model incorporates the classical notion of *bi-moment* (and *bi-shear*) in a fully nonlinear, *geometrically exact* context. Explicit, properly invariant, constitutive equations that generalize those of the linearized theory are developed. The underlying variational formulation of the model is discussed, and computational procedures employing a Galerkin projection are addressed. Numerical examples are presented that illustrate the performance of the formulation.

## 1. INTRODUCTION

We consider a three-dimensional rod model based on a geometrically-exact description of the kinematics of deformation. In addition to finite shear deformation, the model accounts for torsional warping of the cross-sections of the rod and the coupling torsion-bending-warping. Conceptually, the present formulation extends earlier work of Dupuis (1969), Reissner (1973, 1981), Antman (1974), Whitman and DeSilva (1974), Simo (1985) and Simo and Vu-Quoc (1986c) to include the warping distortion of a cross-section in a way that leads to efficient computation. Dupuis (1969) focuses on the linearized stability analysis about the initial configuration. Reissner (1973) generalizes his previous work on the plane problem in Reissner (1972), and extends the classical three-dimensional rod model (cf. Love, 1944) to accommodate the effect of shear deformation. † Analogous models, although phrased in the context of a director theory, were proposed by Whitman and DeSilva (1974), Antman (1974) and Antman and Jordan (1975); these formulations are essentially the two-director Cosserat constrained theories in which the directors are rigid and constrained to remain orthonormal. Subsequently, Parker (1979a,b) performed an asymptotic analysis that provides a justification of the classical rod model (without shear deformation) within the context of the three-dimensional nonlinear theory. Our derivation of the relevant stress resultants, conjugate strain measures, and equations of motion is based on a reduction of the three-dimensional theory by means of a kinematic hypothesis, as in Antman (1976a,b) and Simo (1985).

In recent years, considerable attention has been paid to the problem of coupled bending-torsion-warping of beams. Formulations restricted to the *linear theory* and accounting for the effects of torsion-bending-warping have been proposed by Reissner (1979a,b); including the effect of pretwist by Hodges (1980), Krenk (1983a,b) and Krenk and Gunneskov (1985). Second-order theories accounting for torsion-bending-warping and pretwist have been discussed in Reissner (1983a,b, 1984, 1985). However, as far as we are aware of, the case of finite deformations that include all of the above effects has not been considered heretofore.

† For a historical account of the early development of rod models, see Ericksen and Truesdell (1958).

In what follows, we delineate the basic features and novel aspects of the proposed formulation :

(a) The proposed rod model is fully nonlinear, geometrically exact within the assumed kinematics, and properly invariant under superposed rigid body motions (or isometries). Moreover, complete inertia effects are accounted for in the formulation. The structure of the rate of linear and angular momenta is identical to that arising in rigid body mechanics. In particular, in the limit of an infinitely stiff rod, Euler's equations governing the motion of a rigid body are recovered.

(b) The equation of motion governing the warping of a cross-section is derived with no approximation from the assumed kinematics and the three-dimensional equations of motion. This equation is a complete balance law that relates the bi-moment to the bi-shear and includes the appropriate inertia term.

(c) A class of simple reduced constitutive equations based on small strains is discussed. These constitutive laws are properly invariant under superposed rigid body motions. Coupling effects of torsion, bending and warping are accommodated in the model.

(d) Our earlier computational framework is readily extended to accommodate the effect of warping. Computationally, it amounts simply to the introduction of an additional degree of freedom : There are seven degrees of freedom instead of six.

From a practical standpoint, the proposed formulation and its computational implementation covers a broad range of applications :

(f) It allows the numerical simulation of bifurcation and instability phenomena. An extensive set of numerical examples is given in Simo and Vu-Quoc (1986c, 1988) and Vu-Quoc (1986). The present development includes the effect of torsion-bending-warping which is of particular importance in thin-walled beams with open cross-section and restrained warping.

(g) The dynamic nature of the theory and the exact satisfaction of all the invariance requirements under superposed isometries result in a convenient treatment of flexible beam structures performing large overall motions. This recognition constitutes the basis for our analysis of this class of problems, which encompasses applications as diverse as helicopter blades or rotor blades, flexible robot arms (Simo and Vu-Quoc, 1986a,b), three-dimensional large overall motion of flexible beam structures (Simo and Vu-Quoc, 1986c, 1988), multi-component structures and closed-loop chains undergoing large overall motions, as well as orbiting satellites with flexible appendages and large motion of flexible multibody systems including closed-loop chains (Vu-Quoc and Simo, 1987). A comprehensive overview of this work is given in Vu-Quoc (1986). In particular, we note that :

- (g1) The equations of motion can be referred directly to the *inertial frame*. Extensive coupling in the inertia operator due to Coriolis, centrifugal and inertia due to rotation of the floating frame is completely avoided. For planar motions, for instance, the linearity of the inertia operator with respect to the acceleration simplifies considerably the numerical treatment.
- (g2) Nonlinear geometric effects of special importance in transient dynamic analysis of rotating structures are automatically accounted for in the present formulation. It is numerically demonstrated in Kane *et al.* (1985), and analytically shown in Simo and Vu-Quoc (1987), that the use of geometrically linear theories in such a situation may lead to completely erroneous results.

## 2. KINEMATIC ASSUMPTION. CONFIGURATION SPACE

The following notation is employed throughout the present work.

### 2.1. Basic notation for rotations

Following standard usage, we denote by  $SO(3)$  the rotation group; that is the group of orthogonal matrices  $\Lambda$ , satisfying  $\Lambda\Lambda^T = \mathbf{I}$  and  $\det \Lambda = 1$ . We designate by  $\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes$

$\mathbf{e}_i$ , the identity matrix in  $SO(3)$ . Further, we denote by  $so(3)$  the linear space of skew-symmetric matrices; accordingly,  $\check{\theta} + \check{\theta}^T = \mathbf{0}$ , for all  $\check{\theta} \in so(3)$ . Skew-symmetric matrices  $\check{\theta} \in so(3)$  represent *infinitesimal rotations* with *rotation axis* (or *axial vector*) defined via the standard relation

$$\check{\theta} \mathbf{h} = \boldsymbol{\theta} \times \mathbf{h} \quad \text{for any } \mathbf{h} \in \mathbb{R}^3, \tag{1}$$

where  $\boldsymbol{\theta} \in \mathbb{R}^3$  is the *axial vector* of the skew-symmetric tensor  $\check{\theta}$ , and  $\times$  denotes the ordinary vector product. In coordinates, relative to an orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  of  $\mathbb{R}^3$ , we write†  $\Lambda = \Lambda_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ ,  $\check{\theta} = \check{\theta}_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ , and  $\boldsymbol{\theta} = \theta_i \mathbf{e}_i$ , where  $[\Lambda_{ij}]$  and  $[\check{\theta}_{ij}]$  are orthogonal and skew-symmetric matrices, respectively. In matrix notation we have

$$[\check{\theta}_{ij}] = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}, \quad \{\theta_i\} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}. \tag{2}$$

Next, let  $\Lambda_\tau$  be a time-dependent family of orthogonal matrices, with  $\tau \in [0, T]$  denoting time. Equivalently,  $\Lambda_\tau$  can be viewed as a “curve” on the manifold  $SO(3)$  with parameter  $\tau$ . To compute the tangent to this curve we observe that time differentiation of the orthogonality relation  $\Lambda_\tau \Lambda_\tau^T = \mathbf{I}$  implies that  $\check{\theta}_\tau := [d\Lambda_\tau/d\tau] \Lambda_\tau^T$  is skew-symmetric. Consequently, the tangent field to the curve  $\Lambda_\tau$  is of the form  $d\Lambda_\tau/d\tau := \check{\theta}_\tau \Lambda_\tau$ , with  $\check{\theta}$  skew-symmetric. Hence, given a rotation matrix  $\Lambda \in SO(3)$ , the set

$$T_\Lambda SO(3) := \{\check{\theta} \Lambda \mid \text{for any } \check{\theta} \in so(3)\} \tag{3}$$

is *tangent* to  $SO(3)$  at  $\Lambda$ , and is therefore called the *tangent space*. In particular, at the identity  $\mathbf{I} \in SO(3)$ , the tangent space  $T_\mathbf{I} SO(3)$  is the set of skew-symmetric matrices; i.e.  $so(3)$ . Geometrically  $\check{\theta} \Lambda \in T_\Lambda SO(3)$  represents an infinitesimal rotation (defined by  $\check{\theta}$ ) superposed on (or tangent to) the finite rotation  $\Lambda$ .

2.2. Kinematic description of the beam

We consider a beam which initially occupies a reference configuration denoted by  $B \subset \mathbb{R}^3$ . For simplicity we shall assume that  $B$  corresponds to a *straight beam* with length  $L$ . A convenient parametrization of  $B$  is obtained by introducing an orthogonal frame  $\{O; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$  with coordinates denoted by  $\{X_1, X_2, S\}$  and such that the axis of the beam is initially along  $\mathbf{E}_3$ . The cross-sections of the beam lie therefore in planes parallel to the coordinate plane  $\{\mathbf{E}_1, \mathbf{E}_2\}$  and occupy a region denoted by  $\check{\Omega} \subset \mathbb{R}^2$  with (two-dimensional) boundary denoted by  $\partial\check{\Omega}$ . Consequently,  $B = \check{\Omega} \times [0, L]$  and the position vector  $\mathbf{X}$  of a material point  $X = (X_1, X_2, S)$  in  $B$  is given by

$$\mathbf{X} = X_1 \mathbf{E}_1 + X_2 \mathbf{E}_2 + S \mathbf{E}_3, \quad \text{where } (X_1, X_2) \in \check{\Omega}, \text{ and } S \in [0, L]. \tag{4}$$

† Summation convention on repeated indices is implied. Index in Roman letter takes values in the set  $\{1, 2, 3\}$ , index in Greek letter takes values in the set  $\{1, 2\}$ .

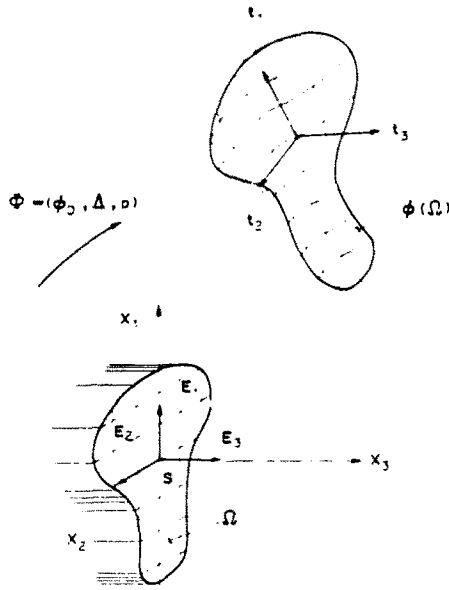


Fig. 1. Geometric description of beam deformation. Definition of material frame  $\{E_i\}$ , spatial frame  $\{e_i := \delta_{ij}E_j\}$ , and (moving) section frame  $\{t_i\}$ .

Let the centroid of a cross-section be at coordinates  $(0, 0, S)$ , and denote by  $A, J_{\alpha\beta}$  and  $J_0$  the area, products of inertia, and polar moment of inertia relative to the centroid. We have†

$$\int_{\Omega} X_i \, d\Omega \equiv 0, \quad A := \int_{\Omega} d\Omega, \quad J_{\alpha\beta} := e_{\alpha\mu}e_{\beta\nu} \int_{\Omega} X_{\mu}X_{\nu} \, d\Omega, \quad J_0 := \int_{\Omega} [X_1^2 + X_2^2] \, d\Omega. \quad (5)$$

We denote by  $\{e_1, e_2, e_3\}$  an *inertial* fixed orthonormal basis in the ambient space  $\mathbb{R}^3$ , such that  $e_i = \delta_{ij}E_j$ . Although the basis vectors  $\{E_i\}$  and  $\{e_i\}$  are chosen to be identical, it proves useful for clarity to maintain the usual distinction between the material basis  $\{E_i\}$  and the spatial basis  $\{e_i\}$ .

With this notation at hand, for the static problem, we specify the deformed configurations of a beam in the three-dimensional ambient space  $\mathbb{R}^3$  as follows:

(i) The line of centroids in the deformed configuration, initially coincident with the interval  $[0, L]$  along  $E_1$ , occupies the position defined by a curve  $\phi_n : [0, L] \rightarrow \mathbb{R}^3$ .

(ii) An arbitrary cross-section of the beam, initially coincident with planes perpendicular to  $E_3$  for values  $S \in [0, L]$ , is assumed to experience a *finite rotation* (and *twist*) about a point  $S$  with position vector  $S = S_2E_2 + SE_3$ , and a superposed out-of-plane *warping*, defined as follows:

(ii.1) Ignoring for the moment warping deformation, a finite rotation (and twist) of the cross-sections is defined by specifying the orientation relative to the basis  $\{E_i\}$  of an orthonormal basis  $\{t_i(S)\}_{i=1,2,3}$  attached to the cross-sections and initially coincident with  $\{E_i\}$ . This is equivalent to prescribing a one-parameter family of orthogonal transformations  $\Lambda : [0, L] \rightarrow SO(3)$  that uniquely define the orientation of the moving frame according to the relations

$$t_i(S) = \Lambda(S)E_i = \Lambda_{ij}(S)e_j. \quad (6)$$

Note that in the *absence* of warping,  $t_3(S)$  is the unit vector normal to the plane of the cross-section in the deformed configuration; see Fig. 1. We also note the relation  $\Lambda(S) = t_i(S) \otimes E_i = \Lambda_{ij}(S)e_i \otimes E_j$ .

†  $e_{\alpha\beta}$  is the permutation symbol defined as  $e_{12} = -e_{21} = 1$  and  $e_{11} = e_{22} = 0$ , and  $\delta_{ij}$  denotes the Kronecker delta.

- (ii.2) Out-of-plane *warping* displacement of each cross-section is then defined by means of an additional displacement *along the unit vector*  $\mathbf{t}_3(S)$  *in the deformed configuration* which is assumed to be given by the product of two functions  $f(X_1, X_2)$  and  $p(S)$ . Here  $f: \Omega \rightarrow \mathbb{R}$  is a prescribed (*given a priori*) warping functions, and  $p: [0, L] \rightarrow \mathbb{R}$  is the (unknown) warping amplitude.†

Now let  $\mathbf{x} = \phi(X_1, X_2, S)$  be the position vector of a material point in the deformed configuration of the beam initially located at  $X = (X_1, X_2, S)$ . According to the preceding discussion, the function  $\phi(X_1, X_2, S)$  is uniquely defined in terms of the functions  $\{\phi_o(S), \Lambda(S), p(S)\}$  by the expression

$$\phi(X_1, X_2, S) = \phi_o(S) + X_2 \mathbf{t}_x(S) + f(X_1, X_2) p(S) \mathbf{t}_3(S). \tag{7}$$

Since a three-dimensional configuration  $\phi$  is uniquely determined by prescribing the triplet of functions  $\Phi \equiv (\phi_o, \Lambda, p)$ , defined on  $[0, L]$  and taking values on  $\mathbb{R}^3 \times SO(3) \times \mathbb{R}$ , one refers to the set

$$C := \{\Phi \equiv (\phi_o, \Lambda, p): [0, L] \rightarrow \mathbb{R}^3 \times SO(3) \times \mathbb{R}\} \tag{8}$$

as the configuration space of the beam. The dynamic case is obtained merely by considering a time-dependent family of configurations, denoted by  $\Phi_t := (\phi_{ot}, \Lambda_t, p_t)$ , which is now a function of  $(S, t) \in [0, L] \times \mathbb{R}_+$ .

This completes our kinematic description of the motion of the beam. The model described above falls within the class of *constrained* Cosserat models in which the directors are constrained to be the orthonormal basis vectors  $\{\mathbf{t}_i\}$ ; see e.g Toupin (1960) and Antman (1972). Note that instead of using three Euler angles to parametrize the orientation of the basis  $\{\mathbf{t}_i\}$  as in Love (1944), Whitman and DeSilva (1974) or Antman (1974), we represent its orientation directly by the orthogonal two-tensor  $\Lambda$  (Dupuis, 1969; Simo, 1985) for simplicity in the exposition of the formulation. However, the use of the orthogonal two-tensor  $\Lambda$  implies a representation of 3-D finite rotations by nine parameters together with three orthonormality constraints (Simo and Vu-Quoc, 1986c). An *optimal* representation of the finite rotations of the beam cross-sections via *quaternions* that avoids the inherent singularity of Euler angles, and at the same time requires only four parameters with one orthonormality constraint, is proposed in Simo and Vu-Quoc (1986b). A possible choice for the warping function  $f: \Omega \rightarrow \mathbb{R}$  is discussed below.

*Remark 2.1. (Linearized kinematics.)* Further insight into the nature of the fully nonlinear kinematic assumption (7) is obtained by examining its linearized version. Set

$$\phi_o = S \mathbf{E}_3 + \varepsilon \mathbf{u}_0 + O(\varepsilon^2), \quad \text{and} \quad \mathbf{t}_t = \mathbf{E}_t + \varepsilon \boldsymbol{\theta} \times \mathbf{E}_t + O(\varepsilon^2), \tag{9}$$

where  $O(\varepsilon^2)/\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . By retaining terms up to the order  $O(\varepsilon)$ , the dynamic case (7) becomes

$$\mathbf{u}(X_1, X_2, S, t) = \mathbf{u}_0(S, t) + \boldsymbol{\theta}(S, t) \times (X_2 \mathbf{E}_x) + f(X_1, X_2) p(S, t) \mathbf{E}_3, \tag{10}$$

where  $\mathbf{u}(X_1, X_2, S, t)$  is the linearized displacement field, of a material point  $X = (X_1, X_2, S)$  at time  $t$ , and  $\mathbf{u}_0(S, t)$  is the *displacement of the line of centroids*. Expression (10) can be written in the following alternative but equivalent form

$$\begin{aligned} u_x(X, t) &= \bar{u}_x(S, t) - e_{2\beta} [X_\beta - S_\beta] \theta_3(S, t) \\ u_3(X, t) &= \bar{u}_3(S, t) - e_{2\beta} X_\beta \theta_2(S, t) + f(X_x) p(S, t), \end{aligned} \tag{11}$$

merely by setting

† We further generalize this description below by assuming that  $f$  is also a function of  $S \in [0, L]$ . Such a generality is necessary to account for the effect of *pretwist*.

$$u_{0\alpha}(S, t) = \tilde{u}_\alpha(S, t) + \theta_\alpha(S, t)e_{z\beta}S_\beta, \quad u_{03}(S, t) = \tilde{u}_3(S, t). \tag{12}$$

Here,  $\tilde{u}(S, t)$  is the displacement of the line of shear centers  $S$  [defined by (14) below]. Clearly, (10) and (11) are equivalent. However, in contrast with (11), (10) has a straight forward extension to the nonlinear range given by (7). ■

*Remark 2.2. (Determination of the warping function.)* The warping function  $f: \Omega \rightarrow \mathbb{R}$  can be determined by considering the Saint-Venant uniform torsion problem for a prismatic body  $B = \tilde{\Omega} \times [0, L]$ . The displacement field  $u(X_1, X_2, S)$  (Sokolnikoff, 1956, p. 113) is obtained from (11) by setting  $\tilde{u}_\alpha = \tilde{u}_3 = 0$  and  $\theta_\alpha = 0$ . The equilibrium equations and the traction-free boundary condition on the lateral surface  $\partial\Omega \times [0, L]$  then result in the classical Neumann problem

$$\Delta f = 0 \text{ in } \Omega, \quad \partial f / \partial v := \nabla f \cdot v = -[(X_\alpha - S_\alpha)E_\alpha \times v] \cdot E_3 \text{ on } \partial\Omega, \tag{13}$$

where  $\nabla f = f_{,\alpha}E_\alpha$ , and  $v = v_\alpha E_\alpha$  denotes the unit vector normal to the boundary  $\partial\Omega$ . Since  $\int_{\partial\Omega} [\partial f / \partial v] d\Gamma = 0$ , standard results in elementary potential theory (cf. Kellogg, 1953) guarantee the existence of a unique solution to problem (13) up to a constant, for prescribed  $(S_1, S_2)$ . Thus, the shear center defined by the constants  $(S_1, S_2)$  and the solution of (13) are uniquely specified by appending the following three additional orthogonality conditions on  $f(X_1, X_2)$ :

$$\int_{\Omega} f(X_1, X_2) d\Omega = \int_{\Omega} X_1 f(X_1, X_2) d\Omega = \int_{\Omega} X_2 f(X_1, X_2) d\Omega = 0. \tag{14}$$

Conditions (14)<sub>2,3</sub> provide a definition of shear center often credited to Trefftz; see e.g. Fung (1969, Appendix I) for an elementary discussion, and Reissner and Tsai (1974) for a different approach. These conditions are also assumed in Vlasov (1961, p. 40). Next, recall the definitions

$$J_S := \int_{\Omega} [(X_1 - S_1)^2 + (X_2 - S_2)^2] d\Omega = J_0 + A[S_1^2 + S_2^2],$$

$$\mathbb{J} := J_S + e_{z\beta} \int_{\Omega} [X_\alpha - S_\alpha] f_{,\beta} d\Omega, \quad \Xi := \int_{\Omega} f^2 d\Omega, \tag{15}$$

where  $\mathbb{J}$  is the Saint-Venant torsion modulus (Sokolnikoff, 1956, p. 112), and  $\Xi$  the warping constant of Vlasov (1961, p. 39). We have the following identities which play an important role in Section 5:

$$\int_{\Omega} f_{,\alpha} d\Omega = -e_{z\beta} S_\beta, \quad \int_{\Omega} (f_{,\alpha}^2 + f_{,\beta}^2) d\Omega = J_S - \mathbb{J},$$

$$\int_{\Omega} e_{z\beta} X_\alpha f_{,\beta} d\Omega = \mathbb{J} - J_0, \quad \int_{\Omega} e_{z\beta} (X_\alpha - S_\alpha) f_{,\beta} d\Omega = \mathbb{J} - J_S, \tag{16}$$

We refer to the Appendix for the proof. ■

*Remark 2.4.* Following Krenk (1983a,b), Krenk and Gunneskov (1985) and Reissner (1985), the effect of pretwist may be accounted for by assuming a warping function now depending on  $S \in [0, L]$  of the form  $f(\xi_1(S), \xi_2(S))$ , which satisfies the following differential equation

$$\frac{df(\xi_1(S), \xi_2(S))}{dS} = \alpha_0(S)e_{\alpha\beta}[\xi_{\alpha}(S)f_{,\beta}(\xi_1(S), \xi_2(S))], \quad (17)$$

where  $\alpha_0(S)$  gives the angle of pretwist. In what follows, we account for the effect of pretwist by considering a general warping function  $f(X_1, X_2, S)$ . ■

2.3. Deformation gradient

We develop in this section a particularly convenient expression for the deformation gradient  $\mathbf{F}(X) = D\phi_t(X)$ , the derivative of the deformation map  $\phi_t$  in (7) with respect to the spatial coordinates  $(X_1, X_2, S)$ . First, in agreement with (3), the spatial and time rates of change of  $\Lambda(S, t)$  are given by

$$\frac{\partial \Lambda(S, t)}{\partial S} = \check{\omega}(S, t)\Lambda(S, t), \quad \frac{\partial \Lambda(S, t)}{\partial t} = \check{\mathbf{w}}(S, t)\Lambda(S, t), \quad (18)$$

where  $\check{\omega}(S, t) \in so(3)$  and  $\check{\mathbf{w}}(S, t) \in so(3)$  are skew-symmetric tensor fields with axial vectors  $\omega(S, t) \in \mathbb{R}^3$  and  $\mathbf{w}(S, t) \in \mathbb{R}^3$ , respectively. In what follows, we will use the customary symbols  $(\cdot)'$  to denote the differentiation  $\partial/\partial S$ , and  $(\dot{\cdot})$  for  $\partial/\partial t$ . It now follows from (7) that

$$\frac{\partial \phi}{\partial X_\alpha} = \mathbf{t}_\alpha + f_{,\alpha} p \mathbf{t}_3, \quad \frac{\partial \phi}{\partial S} = \phi'_\alpha + (X_\alpha - S_\alpha)\omega \times \mathbf{t}_\alpha + (fp)'\mathbf{t}_3 + fp\omega \times \mathbf{t}_3, \quad (19)$$

where  $(\cdot)_{,\alpha}$  denotes partial differentiation with respect to  $X_\alpha$  ( $\alpha = 1, 2$ ). Since  $\Lambda = \mathbf{t}_i \otimes \mathbf{E}_i$ , with the aid of (19), we may express the deformation gradient as

$$\mathbf{F} := \phi_{,i} \otimes \mathbf{E}_i = \Lambda + p \mathbf{t}_1 \otimes \nabla f + [(\phi'_\alpha - \mathbf{t}_\alpha) + \omega \times (\phi - \phi_\alpha) + (fp)'\mathbf{t}_1] \otimes \mathbf{E}_1. \quad (20)$$

Introducing the notation

$$\Gamma := \Lambda'(\phi'_\alpha - \mathbf{t}_\alpha) \equiv \Lambda' \phi'_\alpha - \mathbf{E}_1, \quad \Omega := \Lambda' \omega, \quad (21)$$

we arrive at the following final expression for the deformation gradient

$$\mathbf{F} = \Lambda[\mathbf{I} + p \mathbf{E}_1 \otimes \nabla f + \{\Gamma + \Omega \times \Lambda'(\phi - \phi_\alpha) + (fp)'\mathbf{E}_1\} \otimes \mathbf{E}_1], \quad (22)$$

where  $\Omega = \Lambda' \omega$  is the axial vector of the skew-symmetric tensor  $\check{\Omega} := \Lambda' \Lambda'$ , with components given in the material basis  $\{\mathbf{E}_i\}$ . Expression (22) plays an important role in the developments that follow.

*Remark 2.5.* The above derivation holds for an arbitrary, not necessarily straight, line of centroids. To see this, suppose the line of centroids in the reference configuration is a smooth curve  $\bar{\phi}_\alpha : [a, b] \rightarrow \mathbb{R}^3$ , with length  $L$ . The arc length is then given by

$$S = \bar{S}(\xi) := \int_a^\xi \left\| \frac{d\bar{\phi}_\alpha(\xi)}{d\xi} \right\| d\xi, \quad (23)$$

so that  $S \in [0, L]$  for  $\xi \in [a, b]$  and  $L = \bar{S}(b)$ . Thus, without loss of generality, we may parameterize the reference line of centroids by its arc length  $S$ , so that  $\phi_\alpha(S) \equiv \bar{\phi}_\alpha(\bar{S}^{-1}(S))$ . As for the reference frame  $\{\mathbf{E}_i\}$ , we may choose the Frenet frame and set

$$\mathbf{E}_3 := \phi''_\alpha(S), \quad \mathbf{E}_1 := \frac{\phi''_\alpha(S)}{\|\phi''_\alpha(S)\|}, \quad \mathbf{E}_2 := \mathbf{E}_3 \times \mathbf{E}_1. \quad (24)$$

Expressions (20) and (21) remain valid with

$$\frac{d}{dS} \equiv \frac{1}{\|d\bar{\phi}_n/d\xi\|} \frac{d}{d\xi}$$



3. MECHANICAL POWER. REDUCED CONSTITUTIVE EQUATIONS

We first identify the appropriate stress resultants and stress couples and their conjugate strain measures by deriving the expression for the internal power of the rod. This is accomplished by reduction of the stress power of a 3-D continuum in a manner consistent with the assumed kinematics.

3.1. Stress resultants and stress couples. Conjugate strains

Let  $\mathbf{P}$  denote the first Piola-Kirchhoff stress tensor. Relative to the basis  $\{\mathbf{E}_i\}$  we have the expression

$$\mathbf{P} = \mathbf{T}_x \otimes \mathbf{E}_x + \mathbf{T}_3 \otimes \mathbf{E}_3, \tag{25}$$

Since the base vector  $\mathbf{E}_3$  is normal to a typical cross-section in the reference (undeformed) configuration,  $\mathbf{T}_3 = \mathbf{P}\mathbf{E}_3$  is the stress vector, per unit of reference area  $\Omega$ , acting on that cross-section in the deformed configuration.

Recall that in terms of the first Piola-Kirchhoff stress  $\mathbf{P}$ , the stress power may be expressed as

$$\mathbb{P} := \int_{\Omega \times [0, t]} \mathbf{P} : \dot{\mathbf{F}} \, d\Omega \, dS \tag{26}$$

where  $\mathbf{P} : \dot{\mathbf{F}} := \text{tr}[\mathbf{P}\dot{\mathbf{F}}^T]$ . Our objective is to derive an exact expression for  $\mathbb{P}$  in terms of the stress resultants and stress couples acting on a cross-section. To this end, we make use of (22) along with (18)<sub>2</sub> to obtain

$$\dot{\mathbf{F}} = \check{\mathbf{w}}\mathbf{F} + \Lambda[\dot{\Gamma} + \dot{\Omega} \times \Lambda^T(\phi - \phi_n) + f\dot{p}'\mathbf{E}_3] \otimes \mathbf{E}_3 + \dot{p}\Lambda[\mathbf{E}_3 \otimes \nabla f + f\Omega \times \mathbf{E}_3 + f'\mathbf{E}_3] \otimes \mathbf{E}_3. \tag{27}$$

In deriving expression (27), we have made use of the orthogonality conditions (14) along with the result

$$\frac{\partial}{\partial t}[\Lambda^T(\phi - \phi_n)] = \frac{\partial}{\partial t}[X_x\mathbf{E}_x + f\dot{p}'\mathbf{E}_3] \equiv f\dot{p}'\mathbf{E}_3, \tag{28}$$

which follows at once from (6) and (7). Next, since  $\check{\mathbf{w}}$  is a skew-symmetric tensor and  $\mathbf{P}\mathbf{F}^T/J \equiv \boldsymbol{\sigma}$  is the (symmetric) Cauchy stress tensor, where  $J = \det \mathbf{F}$ , it follows that

$$\mathbf{P} : \check{\mathbf{w}}\mathbf{F} = \text{tr}[\mathbf{P}\mathbf{F}^T\check{\mathbf{w}}^T] = J \text{tr}[\boldsymbol{\sigma}\check{\mathbf{w}}^T] \equiv 0. \tag{29}$$

Hence, by substituting (27) into expression (26) for the stress power  $\mathbb{P}$  and making use of (29), we arrive at the expression for the mechanical power for the rod model

$$\mathbb{P} \equiv \int_{[0, L]} [\mathbf{N} \cdot \dot{\Gamma} + \mathbf{M} \cdot \dot{\Omega} + N_f\dot{p} + M_f\dot{p}'] \, dS, \tag{30}$$

where the stress resultants  $\mathbf{N}$ ,  $\mathbf{M}$ ,  $N_f$  and  $M_f$  have the expressions :



$$\mathbf{N} := \Lambda^T \mathbf{n}, \quad \text{with} \quad \mathbf{n} := \int_{\Omega} \mathbf{T}_3 \, d\Omega, \tag{31a}$$

$$\mathbf{M} := \Lambda^T \mathbf{m}, \quad \text{with} \quad \mathbf{m} := \int_{\Omega} (\boldsymbol{\phi} - \boldsymbol{\phi}_0) \times \mathbf{T}_3 \, d\Omega, \tag{31b}$$

$$N_f := \mathbf{t}_3 \cdot \int_{\Omega} [f_x \mathbf{T}_x + f \mathbf{T}_3 \times \boldsymbol{\omega} + f' \mathbf{T}_3] \, d\Omega, \tag{31c}$$

$$M_f := \mathbf{E}_3 \cdot \left[ \Lambda^T \int_{\Omega} f \mathbf{T}_3 \, d\Omega \right] \equiv \mathbf{t}_3 \cdot \int_{\Omega} f \mathbf{T}_3 \, d\Omega. \tag{31d}$$

*Remark 3.1.* The objects defined by expressions (31) above have a clear physical interpretation.

(a)  $\mathbf{n} = n_i \mathbf{e}_i \equiv N_i \mathbf{t}_i$  and  $\mathbf{m} = m_i \mathbf{e}_i \equiv M_i \mathbf{t}_i$  are the stress resultant and the stress couple acting on a cross-section in the deformed configuration. These two objects are *spatial* vectors expressed either in the fixed spatial basis  $\{\mathbf{e}_i\}$ , or in the cross-section basis  $\{\mathbf{t}_i\}$ .

(b)  $\mathbf{N} = \Lambda^T \mathbf{n}$  and  $\mathbf{M} = \Lambda^T \mathbf{m}$  are *material* vectors obtained by transforming the spatial vectors  $\mathbf{n}$  and  $\mathbf{m}$  back to the reference configuration (a *pull-back* operation using the orthogonal transformation  $\Lambda : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ). Since  $\mathbf{E}_i = \Lambda^T \mathbf{t}_i$ , we have the following coordinate expressions,  $\mathbf{N} = N_i \mathbf{E}_i$  and  $\mathbf{M} = M_i \mathbf{E}_i$ . From the expression of the mechanical power  $\mathbb{P}$  in (30), it can be seen that the strain measures conjugate to  $\mathbf{N}$  and  $\mathbf{M}$  are  $\boldsymbol{\Gamma}$  and  $\boldsymbol{\Omega}$ , respectively. For more detail on the strain measures and their use in computation, we refer readers to the following references: Dupuis (1969), Reissner (1973), Whitman and DeSilva (1974) (component form), Simo (1985) and Simo and Vu-Quoc (1986a) (geometric interpretation and parametrization).

(c)  $N_f$  and  $M_f$  are the counterpart in the finite deformation case of the *bi-shear* and the *bi-moment* in the linear theory of thin-walled beams. In fact, it is shown below that the relation between  $N_f$  and  $M_f$  parallels that of the linear theory. From (30), the bi-shear  $N_f$  is conjugate to the warping amplitude  $p$ ; the bi-moment  $M_f$  is conjugate to the spatial derivative of  $p'$ . ■

*Remark 3.2.* The mechanical power in (30) is expressed in the *material* description. The counterpart of (30) in the *spatial* description takes the form

$$\mathbb{P} \equiv \int_{[0,t,t]} [\mathbf{n} \cdot \overset{\vee}{\boldsymbol{\gamma}} + \mathbf{m} \cdot \overset{\vee}{\boldsymbol{\omega}} + N_f \dot{p} + M_f \dot{p}'] \, dS, \tag{32}$$

where  $\boldsymbol{\omega}$  is the spatial curvature vector defined in (18). Here,  $(\overset{\vee}{\cdot})$  is the *objective* rate measured by an observer fixed in the moving frame  $\{\mathbf{t}_i\}$  and  $\boldsymbol{\gamma}$  is the following spatial vector

$$\boldsymbol{\gamma} := \frac{\partial \boldsymbol{\phi}_0}{\partial S} - \mathbf{t}_3, \quad (\overset{\vee}{\cdot}) := \frac{\partial}{\partial t} (\cdot) - \boldsymbol{\omega} \times (\cdot). \tag{33}$$

It can be seen from (32) that  $\boldsymbol{\gamma}$  is the strain measure conjugate to the spatial stress resultant  $\mathbf{n}$ , and  $\boldsymbol{\omega}$  the curvature vector conjugate to the spatial stress couple  $\mathbf{m}$ . ■

### 3.2. Constitutive equations. Hyperelasticity

Confining our attention to the elastic case and the pure mechanical theory, we postulate the existence of a stored energy function depending on the configuration and its first derivative; i.e.  $\psi = \Psi(S, \boldsymbol{\phi}_0, \Lambda, \boldsymbol{\phi}'_0, \Lambda', p, p')$ , and proceed to enforce the *invariance requirements under superposed rigid body motions*. Let  $t \rightarrow \boldsymbol{\phi}^+ := \mathbf{c}(t) + \mathbf{Q}(t)\boldsymbol{\phi}$ , be a superposed rigid motion where  $\mathbf{c}(t) \in \mathbb{R}^3$  and  $\mathbf{Q}(t) \in SO(3)$ . It follows that

$$\begin{aligned} \phi_o^+ &= Q(t)\phi_o + c(t), & \phi_o'^+ &= Q(t)\phi_o' \\ \Lambda^+ &= Q(t)\Lambda, & \Lambda'^+ &= Q(t)\Lambda', \end{aligned} \tag{34}$$

whereas the fields  $\Gamma, \Omega, p$  and  $p'$  remain unchanged; that is,  $\Gamma^+ = \Gamma, \Omega^+ = \Omega, p^+ = p$  and  $p'^+ = p'$ . Material frame indifference requires that

$$\psi = \tilde{\Psi}(S, Q\phi_o + c, Q\Lambda, Q\phi_o', Q\Lambda', p, p') \equiv \Psi(S, \phi_o, \Lambda, \phi_o', \Lambda', p, p'), \tag{35}$$

for any  $c(t) \in \mathbb{R}^3$ , and any  $Q(t) \in SO(3)$ . In particular: (i) by choosing  $Q(t) = \mathbf{1}$  and  $c(t) = -\phi$  (for fixed  $S \in [0, L]$ ) it follows that  $\Psi$  cannot depend on  $\phi$ ; (ii) by choosing  $Q = \Lambda^T$  (for fixed  $S \in [0, L]$ ) it follows that  $\Lambda$  cannot be an argument in  $\Psi$ , and that the stored energy function takes the form

$$\psi = \Psi(S, \Gamma, \Omega, p, p'). \tag{36}$$

Standard arguments then yield the hyperelastic constitutive equations:

$$\mathbf{N} = \frac{\partial \Psi}{\partial \Gamma}, \quad \mathbf{M} = \frac{\partial \Psi}{\partial \Omega}, \quad N_I = \frac{\partial \Psi}{\partial p}, \quad M_I = \frac{\partial \Psi}{\partial p'}. \tag{37}$$

A concrete example of a properly invariant strain energy function is a quadratic function in the strain measures  $(\Gamma, \Omega, p, p')$ , obtained as a direct generalization from the linear case (see Section 5).

#### 4. MOMENTUM BALANCE LAWS, GOVERNING EQUATIONS

In this section we derive the reduced expressions for the balance laws governing the evolution of the internal forces  $\mathbf{n}, \mathbf{m}, N_I$  and  $M_I$ . Our main result concerns the relation connecting the bi-shear  $N_I$  and the bi-moment  $M_I$ .

##### 4.1. Balance law for the bi-shear and bi-moment

By making use of the exact three dimensional balance of linear momentum equation we derive the exact relation connecting  $N_I$  and  $M_I$ . Let  $\mathbf{B}(X)$  denote the *body force* per unit reference volume acting on the beam, and  $\rho_o(X)$  the density in the reference configuration. Since  $\text{Div } \mathbf{P} = \mathbf{T}_{I,I}$ , the local balance of linear momentum is expressed as

$$\mathbf{T}_{I,I} + \rho_o \mathbf{B} = \rho_o \ddot{\phi}. \tag{38}$$

We use this relation to transform expression (31c) for the bi-shear  $N_I$  as follows. First, note that

$$\begin{aligned} \mathbf{t}_3 \cdot \int_{\Omega} f_{,x} \mathbf{T}_x \, d\Omega &= \mathbf{t}_3 \cdot \int_{\Omega} [(f' \mathbf{T}_x)_{,x} - f' \mathbf{T}_{x,x}] \, d\Omega \\ &= \mathbf{t}_3 \cdot \left[ \int_{\partial\Omega} f' \mathbf{T}_x \nu_x \, dL - \int_{\Omega} f' \mathbf{T}_{x,x} \, d\Omega \right], \end{aligned} \tag{39}$$

where we recall that  $\nu = \nu_x \mathbf{E}_x$  is the normal unit vector to the lateral contour  $\partial\Omega$  of the cross-section  $\Omega$ . We denote by  $\bar{M}_I$  the following scalar resultant

$$\bar{M}_I := \mathbf{t}_3 \cdot \left[ \int_{\partial\Omega} f' \mathbf{T}_x \nu_x \, dL + \int_{\Omega} f' \rho_o \mathbf{B} \, d\Omega \right], \tag{40}$$

which in fact represents the distributed applied bi-moment on the beam. Employing the

local equilibrium relation (38) and the definition (40) in the expression for the bi-shear in (39), we obtain

$$\begin{aligned}
 \mathbf{t}_3 \cdot \int_{\Omega} f_x \mathbf{T}_x \, d\Omega &= \bar{M}_f + \mathbf{t}_3 \cdot \int_{\Omega} f [\mathbf{T}_{3,3} - \rho_o \ddot{\boldsymbol{\phi}} \cdot \mathbf{t}_3] \, d\Omega \\
 &= \bar{M}_f + \int_{\Omega} (f \mathbf{t}_3 \cdot \mathbf{T}_3)_{,3} \, d\Omega - \int_{\Omega} (f \mathbf{t}_3)' \cdot \mathbf{T}_3 \, d\Omega - \int_{\Omega} f \rho_o \ddot{\boldsymbol{\phi}} \cdot \mathbf{t}_3 \, d\Omega \\
 &= \bar{M}_f + M_f' - (\boldsymbol{\omega} \times \mathbf{t}_3) \cdot \int_{\Omega} f \mathbf{T}_3 \, d\Omega - \int_{\Omega} \rho_o f \ddot{\boldsymbol{\phi}} \cdot \mathbf{t}_3 \, d\Omega \quad (41)
 \end{aligned}$$

where we made use of the definition of the bi-moment  $M_f$  in (31d). In view of (31c), eqn (41) implies that

$$M_f' - N_f + \bar{M}_f = \int_{\Omega} \rho_o f \ddot{\boldsymbol{\phi}} \cdot \mathbf{t}_3 \, d\Omega. \quad (42)$$

The right-hand side of the balance eqn (42) may be further reduced by assuming that  $\rho_o(S)$  is independent of the transverse coordinates. Next, making use of (11) and the relation  $\dot{\mathbf{t}}_f = \mathbf{w} \times \mathbf{t}_f$ , there results

$$\begin{aligned}
 \int_{\Omega} \rho_o f \ddot{\boldsymbol{\phi}} \cdot \mathbf{t}_3 \, d\Omega &= \rho_o \int_{\Omega} f^2 \mathbf{t}_3 \cdot \frac{\partial^2}{\partial t^2} (\rho \mathbf{t}_3) \, d\Omega \\
 &= \rho_o \int_{\Omega} f^2 \mathbf{t}_3 \cdot [\ddot{\rho} \mathbf{t}_3 + 2\dot{\rho} \mathbf{w} \times \mathbf{t}_3 + \rho \dot{\mathbf{w}} \times \mathbf{t}_3 + \mathbf{w} \times (\mathbf{w} \times \mathbf{t}_3)] \, d\Omega \\
 &= \rho_o \Xi [\ddot{\rho} - \{(\mathbf{w} \cdot \mathbf{t}_3)^2 - \|\mathbf{w}\|^2\} \rho] \equiv \rho_o \Xi [\ddot{\rho} - \|\mathbf{w} \times \mathbf{t}_3\|^2 \rho]. \quad (43)
 \end{aligned}$$

It remains to establish the equations of balance of linear and angular momenta in terms of stress resultants.

#### 4.2. Summary of the governing equations

Let  $\mathbf{l}(S, t)$  be the *resultant linear momentum*, and let  $\mathbf{h}(S, t)$  be the *resultant angular momentum* relative to  $\boldsymbol{\phi}_o(S, t)$ . By definition, we have

$$\mathbf{l} := \int_{\Omega} \rho_o \dot{\boldsymbol{\phi}} \, d\Omega, \quad \mathbf{h} := \int_{\Omega} \rho_o (\boldsymbol{\phi} - \boldsymbol{\phi}_o) \times \dot{\boldsymbol{\phi}} \, d\Omega. \quad (44)$$

It easily follows from (7) and orthogonality conditions (14) that  $\mathbf{l}(S, t) = \rho_o A \dot{\boldsymbol{\phi}}_o(S, t)$ . Similarly, a more elaborated calculation yields the following result for  $\mathbf{h}(S, t)$ :

$$\mathbf{h}(S, t) = \rho_o(S) \Lambda(S, t) [\mathbf{J}(S) + \rho^2(S, t) \Xi(S) \mathbf{P}_{\mathbf{E}_3}] \mathbf{W}(S, t), \quad (45)$$

where  $\mathbf{W} := \Lambda^T \mathbf{w}$  is the material angular velocity of the cross-section.  $\mathbf{J}$  is the *time-independent* inertia dyadic of the *unwarped* cross-section in the material description, and  $\mathbf{P}_{\mathbf{E}_3}$  is the orthogonal projection parallel to  $\mathbf{E}_3$ , which are given by

$$\begin{aligned}
 \mathbf{J}(S) := \int_{\Omega} X_\alpha X_\beta \, d\Omega [\delta_{\alpha\beta} \mathbf{1} - \mathbf{E}_\alpha \otimes \mathbf{E}_\beta] &\equiv J_{\alpha\beta} \mathbf{E}_\alpha \otimes \mathbf{E}_\beta + J_0 \mathbf{E}_3 \otimes \mathbf{E}_3, \quad \mathbf{P}_{\mathbf{E}_3} := [\mathbf{1} - \mathbf{E}_3 \otimes \mathbf{E}_3].
 \end{aligned} \quad (46)$$

Note that  $[\mathbf{J} + \rho^2 \Xi \mathbf{P}_{\mathbf{E}_3}]$  is the inertia dyadic in the material description, associated with the

warped cross-section, which becomes *time-dependent* due to the presence of the *warping* intensity  $p(S, t)$ . Result (46) can be also expressed in the spatial description.† Time differentiation of (46) then yields

$$\dot{\mathbf{h}} = \rho_o \Lambda [\mathbf{J}\dot{\mathbf{W}} + \mathbf{W} \times \mathbf{J}\dot{\mathbf{W}} + \Xi \{ p^2 [\mathbf{P}_{E_1} \dot{\mathbf{W}} + \mathbf{W} \times \mathbf{P}_{E_1} \dot{\mathbf{W}}] + 2p\dot{p} \mathbf{P}_{E_1} \mathbf{W} \}]. \quad (47)$$

The complete system of equations governing the motion of the rod is summarized in Box I below.

Box I. Governing equations for the rod model. Local form

$\frac{\partial \Lambda(S, t)}{\partial S} = \dot{\omega}(S, t) \Lambda(S, t),$	$\frac{\partial \Lambda(S, t)}{\partial t} = \dot{\omega}(S, t) \Lambda(S, t),$
$\Gamma = \Lambda^r \frac{\partial \phi_o}{\partial S} - \mathbf{E}_3,$	$\Omega = \Lambda^r \omega,$
$\mathbf{n} = \Lambda \frac{\partial \Psi(S, \Gamma, \Omega, p, p')}{\partial \Gamma},$	$\mathbf{m} = \Lambda \frac{\partial \Psi(S, \Gamma, \Omega, p, p')}{\partial \Omega},$
$N_r = \frac{\partial \Psi(S, \Gamma, \Omega, p, p')}{\partial p},$	$M_r = \frac{\partial \Psi(S, \Gamma, \Omega, p, p')}{\partial p'},$
$\frac{\partial \mathbf{n}}{\partial S} + \tilde{\mathbf{n}} = \rho_o \Lambda \ddot{\phi}_o,$	
$\frac{\partial \mathbf{m}}{\partial S} + \frac{\partial \phi_o}{\partial S} \times \mathbf{n} + \tilde{\mathbf{m}} = \dot{\mathbf{h}},$	
$\frac{\partial M_r}{\partial S} - N_r + \tilde{M}_r = \rho_o \Xi [\ddot{p} - \ \mathbf{w} \times \mathbf{t}_3\ ^2 p],$	

*Remark 4.1.* The form of the equations of equilibrium (without the inertia terms) involving  $\mathbf{n}$  and  $\mathbf{m}$  given in Box I are well-known (e.g. Green and Laws, 1966), and have been used in rod models incorporating extensional and shear deformations, as in Reissner (1973) or Antman (1974). These equations can be re-parametrized relative to the arc length in the *deformed* configuration

$$s = \hat{s}(S) := \int_0^S \|\phi'_o\| dS,$$

as in Whitman and DeSilva (1974). The stretch is then  $\partial s / \partial S = \|\phi'_o\|$ . Further, if shear deformation is neglected,  $\mathbf{t}_3 = \phi'_o / \|\phi'_o\|$  and the equilibrium equations in Box I now parameterized in terms of the current arc length  $s$  reduce to

$$\frac{\partial \mathbf{n}}{\partial s} + \tilde{\mathbf{n}} = 0, \quad \frac{\partial \mathbf{m}}{\partial s} + \mathbf{t}_3 \times \mathbf{n} + \tilde{\mathbf{m}} = 0, \quad (48)$$

where  $\tilde{\mathbf{n}} := \tilde{\mathbf{n}} / \|\phi'_o\|$  and  $\tilde{\mathbf{m}} := \tilde{\mathbf{m}} / \|\phi'_o\|$  are the distributed applied force and couple per unit *deformed* length. Equations (48) are in the form given in Ericksen and Truesdell (1958).‡ By defining the material curvature vector  $\mathbf{K}(s)$  such that

† Set  $\mathbf{j}(S, t) := \Lambda(S, t) \mathbf{J}(S) \Lambda^T(S, t)$  and  $\mathbf{P}_{E_1}(S, t) = \Lambda(S, t) \mathbf{P}_{E_1}(S) \Lambda^T(S, t)$ , and recall that  $\mathbf{w} = \Lambda \dot{\mathbf{W}}$ , so that  $\mathbf{h}(S, t) = [\mathbf{j}(S, t) + p^2(S, t) \Xi \mathbf{P}_{E_1}(S, t)] \mathbf{w}(S, t)$ . Note that in the spatial description,  $\mathbf{j}(S, t) \equiv J_{, \mu} [\delta_{3\mu} \mathbf{1} - \mathbf{t}_3 \otimes \mathbf{t}_3]$  and  $\mathbf{P}_{E_1}(S, t) = [\mathbf{1} - \mathbf{t}_3 \otimes \mathbf{t}_3]$  are now *time-dependent*.

‡ See equations (22.1) and (22.2), p. 316, of Ericksen and Truesdell (1958).

$$\frac{\partial \Lambda}{\partial s} = \|\phi_s'\| \Lambda \check{\Omega} = \Lambda \check{K}, \tag{49}$$

where  $\mathbf{K}(s(S)) := \|\phi_s'(S)\| \Omega(S)$ , in terms of the material stress resultant  $\mathbf{N}$  and stress couple  $\mathbf{M}$ , (48) become

$$\frac{\partial \mathbf{N}}{\partial s} + \mathbf{K} \times \mathbf{N} + \check{\mathbf{N}} = 0, \quad \frac{\partial \mathbf{M}}{\partial s} + \mathbf{K} \times \mathbf{M} + \mathbf{E}_3 \times \mathbf{N} + \check{\mathbf{M}} = 0, \tag{50}$$

where  $\check{\mathbf{N}} := \Lambda^T \check{\mathbf{n}}$ ,  $\check{\mathbf{M}} := \Lambda^T \check{\mathbf{m}}$ . The equilibrium equations (50) are (in component form) those given in Love (1944) for a thin rod without shear and warping deformation. † ■

*Remark 4.2.* The momentum eqns (42)–(43) involving the bi-moment and bi-shear are *exact* and were derived above in the context of the *fully nonlinear theory*. Remarkably, the static version of this equation is identical to that of the linear theory; see e.g. Vlasov (1961). Reissner (1983b, 1984) derived the static version of (42) in the context of a *second-order geometrically nonlinear theory*. ■

*Remark 4.3.* For the dynamic case, Green and Laws (1966) considered the mass to be distributed on the representative curve modeling the rod in their two-director theory; hence there is no inertia term in the equation for the resultant couple, i.e. rotatory inertia is neglected. Whitman and DeSilva (1969) derived a complete set of equations of motion for their Cosserat curve that include the rotatory inertia. Antman and Liu (1979) also consider both translational and rotatory inertia effects in their study of traveling waves in hyperelastic rods. A complete analysis of the underlying Hamiltonian structure is given in Simo *et al.* (1988). ■

### 5. A CLASS OF REDUCED ELASTIC CONSTITUTIVE EQUATIONS

We discuss below properly invariant constitutive equations in terms of stress resultants and stress couples of the form (37). These constitutive relations are extensions of results obtained in the infinitesimal theory to the finite deformation range. Although these relations could be postulated at the outset, the final form can be motivated as follows.

#### 5.1. Elementary motivation. Infinitesimal strains

Let  $\mathbf{H}$  be a rank-two material tensor defined as  $\mathbf{H} := \Lambda^T \mathbf{F} - \mathbf{1}$ . From (22), we have

$$\mathbf{H} = p \mathbf{E}_3 \otimes \nabla f + [\Gamma + \Omega \times \Lambda^T (\phi - \phi_n) + f p' \mathbf{E}_3] \otimes \mathbf{E}_3. \tag{51}$$

We introduce the assumption of *small strains*, although arbitrarily large displacements and rotations, by considering a small parameter  $\varepsilon > 0$  and assuming that  $\|\mathbf{H}\| = O(\varepsilon)$ , where  $O(\varepsilon)/\varepsilon$  tends to a constant as  $\varepsilon \rightarrow 0$ . Accordingly, we set

$$\mathbf{F}_\varepsilon = \Lambda[\mathbf{1} + \varepsilon \mathbf{H}] \Rightarrow \mathbf{E}_\varepsilon = \varepsilon \mathbf{H}^S + O(\varepsilon^2), \tag{52}$$

where  $\mathbf{E}_\varepsilon := \frac{1}{2}[\mathbf{F}_\varepsilon^T \mathbf{F}_\varepsilon - \mathbf{1}]$  is the Lagrangian strain tensor, and  $\mathbf{H}^S = \frac{1}{2}[\mathbf{H} + \mathbf{H}^T]$  denotes the symmetric part of  $\mathbf{H}$ . In addition, let  $\sigma$  be the Cauchy stress tensor. We assume that  $\|\sigma\| = O(\varepsilon)$ , so that the second Piola-Kirchhoff tensor  $\mathbf{S}_\varepsilon$  becomes

$$\mathbf{S}_\varepsilon = \varepsilon \Sigma + O(\varepsilon^2), \quad \text{where } \Sigma := \Lambda^T \sigma \Lambda. \tag{53}$$

In addition, because of (53) we have

† Equations (10) and (11) on pp. 387–388 of Love (1944).

$$\Lambda^T \mathbf{P}_\epsilon = (\det \mathbf{F}_\epsilon) \Lambda^T \boldsymbol{\sigma} \mathbf{F}_\epsilon^{-T} = \epsilon \boldsymbol{\Sigma} + \mathbf{O}(\epsilon^2). \tag{54}$$

In what follows, we shall restrict our attention to the case of small strains by considering first order  $O(\epsilon)$  constitutive relations. No restriction, however, is placed on the displacement field  $\phi_\nu(S, t)$  or on the rotation field  $\Lambda(S, t)$ . Accordingly, we may postulate a *linear* isotropic relation between  $\mathbf{S}_\epsilon$  and  $\mathbf{E}_\epsilon$  with the same structure as in the linear theory; i.e. the so-called Saint-Venant/Kirchhoff material, see Gurtin (1981) or Marsden and Hughes (1983). Because of relations (52) and (53), this is equivalent, within order  $O(\epsilon^2)$ , to postulating a relation

$$\Sigma_{\alpha\beta} = [\lambda \delta_{\alpha\beta} \delta_{\rho\theta} + 2G \delta_{\alpha\rho} \delta_{\beta\theta}] H_{\rho\theta}^S, \tag{55}$$

where  $\boldsymbol{\Sigma} = \Sigma_{\alpha\beta} \mathbf{E}_\alpha \otimes \mathbf{E}_\beta$ ,  $\mathbf{H}^S = H_{\rho\theta}^S \mathbf{E}_\rho \otimes \mathbf{E}_\theta$ ,  $\lambda > 0$  and  $G > 0$  denotes the Lamé's constants. Although (55) are linearized relations, we emphasize that they are properly invariant under superposed isometries. To proceed further, we note from (54) that, to first order,  $\Lambda^T \mathbf{T}_3 = \Sigma_{I3} \mathbf{E}_I$ . Furthermore, as in elementary derivations of the linear theory, assume that

$$\Sigma_{\alpha 3} = 2GH_{\alpha 3}^S, \quad \Sigma_{33} = EH_{33}^S, \tag{56}$$

where  $G$  denotes the shear modulus and  $E$  the Young's modulus. Finally, by observing from (51) that

$$2H_{\alpha 3}^S = \mathbf{E}_\alpha \cdot [\boldsymbol{\Gamma} + \boldsymbol{\Omega} \times (X_\beta \mathbf{E}_\beta + fp \mathbf{E}_3) + p \nabla f], \quad H_{33}^S = \mathbf{E}_3 \cdot [\boldsymbol{\Gamma} + \boldsymbol{\Omega} \times X_\alpha \mathbf{E}_\alpha + fp' \mathbf{E}_3], \tag{57}$$

and setting  $\Gamma_\alpha := \boldsymbol{\Gamma} \cdot \mathbf{E}_\alpha$ , we arrive at the following constitutive relations.

*Stress resultant N.* By making use of the orthogonality relations (14), the condition (2)<sub>1</sub> and the identity (16)<sub>1</sub>, from definition (31a) we obtain the relation

$$\begin{aligned} \mathbf{N} &= \Lambda^T \int_\Omega \mathbf{T}_3 \, d\Omega = \int_\Omega \Sigma_{I3} \mathbf{E}_I \, d\Omega = \left[ GA \Gamma_\alpha + Gp \int_\Omega f_{,\alpha} \, d\Omega \right] \mathbf{E}_\alpha + EA \Gamma_3 \mathbf{E}_3, \\ &= [GA \Gamma_\alpha - Gc_{\alpha\beta} S_\beta p] \mathbf{E}_\alpha + EA \Gamma_3 \mathbf{E}_3. \end{aligned} \tag{58}$$

*Stress couple M.* Substitution of (57), conditions (14) and (5)<sub>1</sub>, and definition (5)<sub>3</sub> into definition (31b) of the *material stress couples*  $\mathbf{M}$  relative to the centroid, results in the following constitutive equation

$$\begin{aligned} \mathbf{M} &= \Lambda^T \int_\Omega (\boldsymbol{\phi} - \boldsymbol{\phi}_0) \times \mathbf{T}_3 \, d\Omega = \int_\Omega (X_\alpha \mathbf{E}_\alpha + pf \mathbf{E}_3) \times [2GH_{\beta 3}^S \mathbf{E}_\beta + EH_{33}^S \mathbf{E}_3] \, d\Omega \\ &= EJ_{\alpha\beta} \Omega_\beta \mathbf{E}_\alpha + G \left[ J_0 \Omega_3 + pc_{\alpha\beta} \int_\Omega X_\alpha f_{,\beta} \, d\Omega \right] \mathbf{E}_3, \end{aligned} \tag{59}$$

where  $J_{\alpha\beta}$  and  $J_0$  are the components of the inertia dyadic  $\mathbf{J}$  as defined in (46). Note that in deriving (59) we have neglected nonlinear terms in  $\rho^2$ . Use of the identity (16)<sub>3</sub> finally yields

$$\mathbf{M} = EJ_{\alpha\beta} \Omega_\beta \mathbf{E}_\alpha + G[\mathbb{J}p + J_0(\Omega_3 - p)] \mathbf{E}_3. \tag{60}$$

*Bi-shear N<sub>I</sub> and bi-moment M<sub>I</sub>.* First note that the second term in the expression for the bi-shear  $N_I$  as given in (31c) is of the order  $O(\epsilon^2)$  and can be neglected consistently with relation (54). Hence, making use of (14), (5) and identities (16) we obtain

$$\begin{aligned}
 N_f &= \mathbf{t}_3 \cdot \int_{\Omega} f_{,x} \mathbf{T}_x \, d\Omega = E_3 \cdot \int_{\Omega} f_{,x} \Lambda^T \mathbf{T}_x \, d\Omega = E_3 \cdot \int_{\Omega} f_{,x} \Sigma_{I\alpha} \mathbf{E}_I \, d\Omega = 2G \int_{\Omega} f_{,x} H_{3\alpha}^S \, d\Omega \\
 &= G \left[ \Gamma_x \int_{\Omega} f_{,x} \, d\Omega + \Omega_3 e_{\alpha\beta} \int_{\Omega} X_{\alpha} f_{,\beta} \, d\Omega + \rho \int_{\Omega} f_{,x}^2 \, d\Omega \right], \\
 &= G[e_{\alpha\beta} S_{\alpha} \Gamma_{\beta} + (\mathbb{J} - J_0)\Omega_3 + (J_S - \mathbb{J})\rho]. \tag{61}
 \end{aligned}$$

Finally, definition (31d) for  $M_f$  and (54) along with (56), (57) and (14) yield the classical relation

$$M_f = E \int_{\Omega} f H_{33}^S \, d\Omega = E \Xi p'. \tag{62}$$

The preceding constitutive relations can be written in matrix notation as

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ M_1 \\ M_2 \\ M_3 \\ N_f \\ M_f \end{Bmatrix} = \begin{bmatrix} GA & \cdot & \cdot & \cdot & \cdot & \cdot & -GS_2 & \cdot \\ & GA & \cdot & \cdot & \cdot & \cdot & GS_1 & \cdot \\ & & EA & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & EJ_{11} & EJ_{12} & \cdot & \cdot & \cdot \\ & & & & EJ_{22} & \cdot & \cdot & \cdot \\ & & & & & GJ_0 & G(\mathbb{J} - J_0) & \cdot \\ & SYM. & & & & & G(J_S - \mathbb{J}) & \cdot \\ & & & & & & & E\Xi \end{bmatrix} \begin{Bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ p \\ p' \end{Bmatrix}. \tag{63}$$

We note that the matrix of *material elastic moduli* is constant and *symmetric*; hence, the strain energy function (36) becomes a quadratic functional in the (material) strain variables  $\{\Gamma, \Omega, p, p'\}$ .

*Remark 5.1.* In the case of uniform (Saint-Venant) torsion, one has  $p \equiv \Omega_3$ , and constitutive eqn (59) reduces to the classical Saint-Venant solution  $M_3 = G\mathbb{J}\Omega_3$ . ■

*Remark 5.2.* It is well known that direct integration of the local stress-strain relations based on assumed beam kinematics overestimates the shear stiffness, which is often corrected by introducing the so-called *shear coefficient*. Several procedures have been proposed to estimate this coefficient; see e.g. Cowper (1966). Similarly, the Saint-Venant torsional constant  $\mathbb{J}$  defined in (16) is often used in lieu of the polar moment of inertia  $J_0$  (which is the constant that arises in elementary beam theory without warping but including torsional effects).

For thin-walled beams, Krenk and Gunneskov (1985) propose the use of a symmetric tensor of generalized shear-corrected areas  $A_{\alpha\beta}$  as a means of improving the shear stresses distribution. In the present context this procedure can be readily implemented by assuming an appropriate stored energy function. For instance, according to Krenk and Gunneskov (1985),

$$\Psi = \frac{1}{2} [GA_{\alpha\beta} \Gamma_{\alpha} \Gamma_{\beta} + EA\Gamma_3^2 + EJ_{\alpha\beta} \Omega_{\alpha} \Omega_{\beta} + G\mathbb{J}\Omega_3^2 G(J_0 - \mathbb{J})(\Omega_3 - p)^2 + E\Xi(p')^2], \tag{64}$$

where  $\Gamma_{\alpha}$  are modified shear strains. If we set  $\Gamma_{\alpha} := \Gamma_{\alpha} - e_{\alpha\beta} S_{\beta} p$ , for  $\alpha = 1, 2$ , use of the general expressions (37) yields constitutive equations with essentially the same structure as (63). ■.

† Note that Whitman and DeSilva (1974) used this type of constitutive equation but with  $GJ_0$  for torsional stiffness instead of  $G\mathbb{J}$  [see eqn (4.3) of this reference].

6. VARIATIONAL FORMULATION. LINEARIZATION

In this section we construct the weak form (virtual work expression) of the local field equations summarized in Box 1, and discuss the variational structure underlying this model. Subsequently, we address the linearization of the weak form and derive the local form of the linearized equations. These linearized equations play a crucial role in an iterative solution procedure of the Newton type (method of tangents) performed on *function spaces*, in linearized stability and bifurcation analyses (e.g. Dupuis, 1969; Antman and Kenney, 1981), and in the study of the small deformations superimposed on finite deformations (Green *et al.*, 1968).

6.1. *Admissible variations. Weak form of the equilibrium equations*

The equilibrium equations in Box 1, without the inertia force, can be rewritten in the following compact form

$$\mathbb{B}^*(\Phi)\mathbf{r} - \bar{\mathbf{f}} = 0, \quad \text{in } [0, L], \tag{65}$$

where  $\mathbf{r}$  is the *vector of stress resultants*,  $\bar{\mathbf{f}}$  is the applied *resultant force*, and  $\mathbf{r} \rightarrow \mathbb{B}^*\mathbf{r}$  is a *configuration-dependent* differential operator defined as

$$\mathbb{B}^*(\Phi)\mathbf{r} := \begin{Bmatrix} -\mathbf{n} \\ -\mathbf{m} - \phi'_s \times \mathbf{n} \\ -M'_j + N_j \end{Bmatrix}_{7 \times 1}, \quad \mathbf{r} := \begin{Bmatrix} \mathbf{n} \\ \mathbf{m} \\ N_j \\ M_j \end{Bmatrix}_{8 \times 1}, \quad \bar{\mathbf{f}} := \begin{Bmatrix} \bar{\mathbf{n}} \\ \bar{\mathbf{m}} \\ \bar{M}_j \end{Bmatrix}_{7 \times 1} \tag{66}$$

To define the virtual work expression associated with (65) we introduce the space of kinematically admissible variations in the standard manner as follows.

*Admissible variations.* Assume for simplicity pure displacement boundary conditions, and let

$$V := \{ \eta := (\mathbf{u}, \theta, q) : [0, L] \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \mid \eta|_{s=0,L} = (0, 0, 0) \}. \tag{67}$$

Given a configuration  $\Phi \equiv (\phi, \Lambda, p) \in C$ , any element  $\eta = (\mathbf{u}, \theta, q) \in V$  uniquely defines curve of configurations  $\tau \rightarrow \Phi_\tau^? \in C$ , which starts at  $\Phi \in C$  and is given by

$$\Phi_\tau^? := (\phi + \tau\mathbf{u}, \exp[\tau\check{\theta}]\Lambda, p + \tau q) \in C, \quad \text{with } \Phi_\tau^?|_{\tau=0} = \Phi. \tag{68}$$

Here,  $\exp[\tau\check{\theta}] \in SO(3)$  denotes the *orthogonal matrix* obtained by exponentiating the skew-symmetric matrix  $\tau\check{\theta}$ . By definition,  $d[\Phi_\tau^?/d\tau]|_{\tau=0}$  is called a *variation* of  $\Phi \in C$  in the direction  $\eta \in V$ . By differentiating (6.4) with respect to  $\tau$  it follows that the *tangent space of variations* at  $\Phi = (\phi_s, \Lambda, p)$ , which is denoted by  $T_\Phi C$ , is given by

$$T_\Phi C := \{ (\mathbf{u}, \check{\theta}\Lambda, q) \mid (\mathbf{u}, \theta, q) \in V \}. \tag{69}$$

Clearly, these are a one-to-one correspondence between  $T_\Phi C$  and  $V$ . With a slight abuse of notation, in what follows, an element of  $V$  is often referred to simply as a *variation*.

By multiplying (65) by a variation  $\eta \in V$  and integrating by parts, we obtain the following virtual work expression; a function  $G : C \times V \rightarrow \mathbb{R}$  given by

$$G(\Phi, \eta) := \int_0^L \mathbb{B}(\Phi)\eta \cdot \mathbf{r} \, dS - \int_0^L \eta \cdot \bar{\mathbf{f}} \, dS = 0, \quad \text{for all } \eta \in V, \tag{70}$$

where  $\mathbb{B}(\Phi)$  is the differential operator defined as



$$\boldsymbol{\eta} \equiv (\mathbf{u}, \boldsymbol{\theta}, q) \in V \rightarrow \mathbb{B}(\Phi)\boldsymbol{\eta} := \left\{ \begin{array}{c} \mathbf{u}' + \boldsymbol{\phi}'_o \times \boldsymbol{\theta} \\ \boldsymbol{\theta}' \\ q \\ q' \end{array} \right\}_{8 \times 1} \quad (71)$$

The vector of resultants  $\mathbf{r}$  is a function of the configuration through the elastic relations (37). We introduce the notation

$$\mathbf{E}(\Phi) := \left\{ \begin{array}{c} \Gamma \\ \Omega \\ p \\ p' \end{array} \right\}_{8 \times 1}, \quad \mathbf{R} := \left\{ \begin{array}{c} \mathbf{N} \\ \mathbf{M} \\ N_f \\ M_f \end{array} \right\}_{8 \times 1}, \quad \boldsymbol{\Pi} = \begin{bmatrix} \Lambda & \cdot & \cdot \\ & \Lambda & \cdot \\ & & \mathbf{1}_{2 \times 2} \end{bmatrix}_{8 \times 8}, \quad (72)$$

so that constitutive eqns (37) and the corresponding elasticity tensors then take the following compact form

$$\begin{aligned} \mathbf{R} &= \frac{\partial \Psi(\mathbf{S}, \mathbf{E})}{\partial \mathbf{E}}, & \mathbf{r} &= \boldsymbol{\Pi} \mathbf{R}, \\ \mathbf{C} &= \frac{\partial^2 \Psi(\mathbf{S}, \mathbf{E})}{\partial \mathbf{E}^2}, & \mathbf{c} &= \boldsymbol{\Pi} \mathbf{C} \boldsymbol{\Pi}^T. \end{aligned} \quad (73)$$

The quadratic constitutive relation considered in Section 5 corresponds to the case  $\mathbf{C} = \text{Const.}$

*Remark 6.1.* The constitutive eqns (73) could be phrased in *weak form* as follows. Let  $\chi(\mathbf{S}, \mathbf{R}) := -\Psi(\mathbf{S}, \mathbf{E}) + \mathbf{R} \cdot \mathbf{E}$  be the Legendre transformation of  $\Psi(\mathbf{S}, \mathbf{E})$ . Define

$$H(\Phi, \mathbf{R}; \delta \mathbf{R}) := \int_0^L \delta \mathbf{R} \cdot \left[ -\frac{\partial \chi(\mathbf{S}, \mathbf{R})}{\partial \mathbf{R}} + \mathbf{E}(\Phi) \right] dS. \quad (74)$$

Then  $H(\Phi, \mathbf{R}; \delta \mathbf{R}) = 0$  for arbitrary  $\delta \mathbf{R}$  is the Hellinger–Reissner weak form of the constitutive equations. This form plays a central role in recent finite element formulations of the incompressible problem (Franca and Hughes, 1988) and plates and shells (Simo and Fox, 1989; Simo *et al.*, 1989). ■

6.2. *Linearization. Potential operators. Newton’s method*

First, we record the expression for the linearization of the strain measures  $\mathbf{E}(\Phi)$  at a configuration  $\Phi$  in the direction a variation  $\boldsymbol{\xi} \in V$ , which is denoted by  $D\mathbf{E}(\Phi) \cdot \boldsymbol{\xi}$ . Given  $\boldsymbol{\xi} \in V$ , consider the curve of configurations  $\Phi_t^{\boldsymbol{\xi}}$  defined as in (67). Using the chain rule, by the directional derivative formula one has the result (Simo and Vu-Quoc, 1986c),

$$D\mathbf{E}(\Phi) \cdot \boldsymbol{\xi} := \left. \frac{d}{dt} \right|_{t=0} \mathbf{E}(\Phi_t^{\boldsymbol{\xi}}) = \boldsymbol{\Pi} \mathbb{B}(\Phi) \boldsymbol{\xi}, \quad \text{for any } \boldsymbol{\xi} \in V, \quad (75)$$

where  $\mathbb{B}(\Phi)$  is defined by (71). The linearization of the weak form  $G(\Phi, \boldsymbol{\eta})$  at configuration  $\Phi \in C$  in the direction of a variation  $\boldsymbol{\xi} = (\bar{\mathbf{u}}, \bar{\boldsymbol{\theta}}, \bar{q}) \in V$  is a bilinear form on  $V$ , which is denoted by  $DG(\Phi, \boldsymbol{\eta}) \cdot \boldsymbol{\xi}$ , and is given by

$$DG(\Phi, \boldsymbol{\eta}) \cdot \boldsymbol{\xi} = \left. \frac{d}{d\tau} \right|_{\tau=0} G(\Phi_t^{\boldsymbol{\xi}}, \boldsymbol{\eta}), \quad (76)$$

where, as in (67),  $[d\Phi_t^{\boldsymbol{\xi}}/d\tau]_{\tau=0} = (\bar{\mathbf{u}}, \bar{\boldsymbol{\theta}}, \bar{q})$ . Making use of (75) and the chain rule, (76) results in the following expression

$$DG(\Phi, \eta) \cdot \xi = \int_0^L \mathbb{B}(\Phi)\eta \cdot \mathbf{c}\mathbb{B}(\Phi)\xi \, dS + \int_0^L \mathbb{L}\eta \cdot \mathbf{b}\mathbb{L}\xi \, dS. \tag{77}$$

Here,  $\mathbf{c}$  is the spatial elasticity tensor defined in (73),  $\mathbf{b}$  is the so-called *initial stress matrix*, which in the present model is a  $(9 \times 9)$  matrix, and  $\mathbb{L}$  is a matrix differential operator with the following expressions

$$\mathbf{b} := \begin{bmatrix} \mathbf{O}_{3 \times 3} & \mathbf{O}_{3 \times 3} & -\check{\mathbf{n}} \\ \mathbf{O}_{3 \times 3} & \mathbf{O}_{3 \times 3} & -\check{\mathbf{m}} \\ \check{\mathbf{n}} & \mathbf{O}_{3 \times 3} & [\mathbf{n} \otimes \phi'_\sigma - (\phi'_\sigma \cdot \mathbf{n})\mathbf{I}]_{9 \times 9} \end{bmatrix}, \quad \eta \equiv (\mathbf{u}, \theta, q) \rightarrow \mathbb{L}\eta := \begin{Bmatrix} \mathbf{u}' \\ \theta' \\ \theta \end{Bmatrix}. \tag{78}$$

*Remark 6.2.* Formally, if enough smoothness is assumed, the weak formulation as given by (70) implies the local form (65) of the equilibrium equations. This is a standard result in the calculus of variation which is obtained through integration by parts; see e.g. Mikhlin (1970). Similarly, the *local form* of the *linearized* equilibrium equations is formally obtained from (77) by integration by parts. In fact, from (77) there follows

$$DG(\Phi, \eta) \cdot \xi = \int_0^L \eta \cdot \{ \mathbb{B}^*(\Phi)[\mathbf{c}\mathbb{B}(\Phi)\xi] + \mathbb{L}^*[\mathbf{b}\mathbb{L}\xi] \} \, dS \tag{79}$$

where  $\mathbb{L}^*$  is the adjoint operator of  $\mathbb{L}$ . In matrix notation, one has the expressions

$$\mathbb{L} := \begin{bmatrix} \frac{d}{dS} \mathbf{I}_3 & \mathbf{O}_{3 \times 3} & \mathbf{O}_{3 \times 1} \\ \mathbf{O}_{3 \times 3} & \frac{d}{dS} \mathbf{I}_3 & \mathbf{O}_{3 \times 1} \\ \mathbf{O}_{3 \times 1} & \mathbf{I}_3 & \mathbf{O}_{3 \times 1} \end{bmatrix}_{9 \times 7}, \quad \mathbb{L}^* := \begin{bmatrix} -\frac{d}{dS} \mathbf{I}_3 & \mathbf{O}_{3 \times 3} & \mathbf{O}_{3 \times 3} \\ \mathbf{O}_{3 \times 3} & -\frac{d}{dS} \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{O}_{3 \times 3} & \mathbf{O}_{3 \times 3} & \mathbf{O}_{3 \times 3} \end{bmatrix}_{7 \times 9}. \tag{80}$$

The operator within brackets in (79) is referred to as the *local tangent* operator at configuration  $\Phi \in C$ . We use the notation

$$\mathbb{K}(\Phi) = \mathbb{B}^*(\Phi)[\mathbf{c}\mathbb{B}(\Phi)] + \mathbb{L}^*[\mathbf{b}\mathbb{L}]. \tag{81}$$

*Remark 6.3.* The second term in (77), or the term  $\mathbb{L}^*[\mathbf{b}\mathbb{L}]$  in (81), is referred to as the *geometric part* of the tangent operator. Since  $\mathbf{b}$  is *generally non-symmetric*, see (78), it follows from (77) that  $DG(\Phi, \eta) \cdot \xi \neq DG(\Phi, \xi) \cdot \eta$ . Equivalently, for arbitrary  $\Phi \in C$ , the tangent operator  $\mathbb{K}(\Phi)$  defined by (81) is generally non-self-adjoint. However, as shown in Simo and Vu-Quoc (1986c), symmetry of the linearized weak form or, equivalently, self-adjointness of the local tangent operator, is recovered at an *equilibrium configuration*, provided the loading is conservative. Away from equilibrium, a proper definition of the Hessian also leads to a *symmetric* tangent operator (see Simo, 1990). ■

*Remark 6.4.* The linearized problem plays a central role in the iterative solution of nonlinear boundary value problems. The basic tool here is Kantorovich's extension of Newton's method to nonlinear operators defined in Banach spaces of functions, e.g. Vainberg (1964). In a numerical analysis context, the procedure is often referred to as the *incremental method*. We refer to Bernadou *et al.* (1984) for a recent mathematical analysis of the method within the framework of nonlinear elasticity. The formulation and application of the method to three-dimensional geometrically exact rod models is considered in detail in Simo and Vu-Quoc (1986c) in the context of the static problem, and in Simo and Vu-Quoc (1988) in the context of the dynamic problem. The results in this section enable one to extend this methodology to the situation in which torsion-warping is taken into account with essentially no modification. Hence, further details are omitted. ■

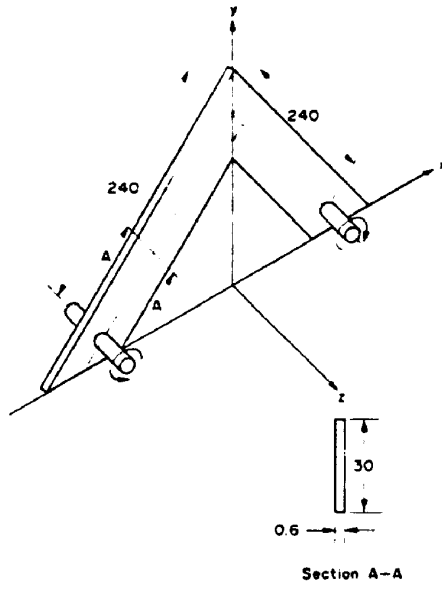


Fig. 2. Right angle frame. Problem description.

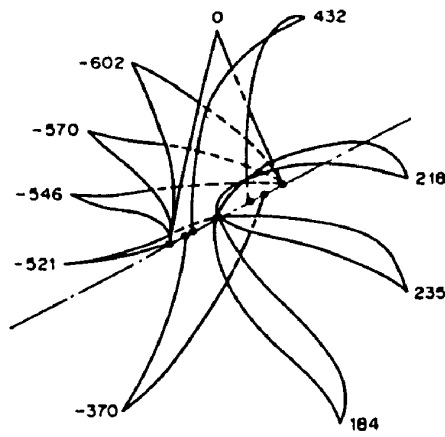


Fig. 3. Right angle frame. Perspective view of first revolution of snap-through motion.

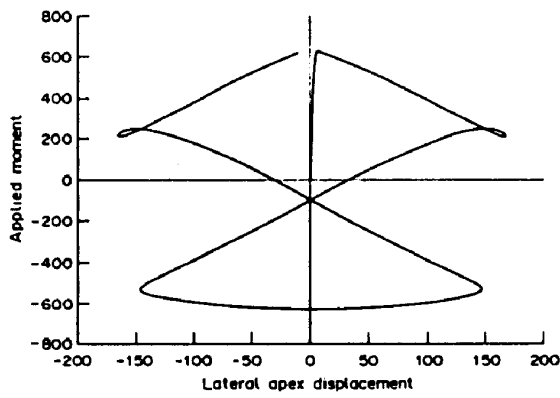


Fig. 4. Right angle frame. Applied moment versus lateral displacement of apex.

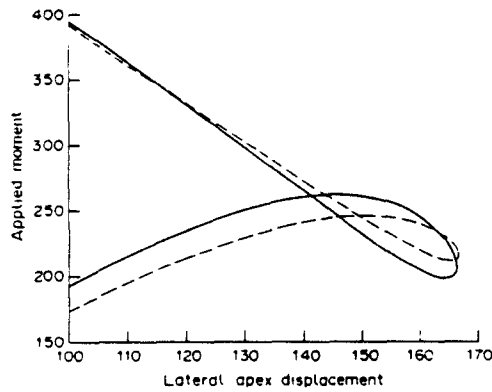


Fig. 5. Right angle frame. Magnified portion of the applied moment versus apex lateral displacement diagram. Comparison of formulations with warping (*solid line*) and without warping (*dotted line*).

### 6.3. Numerical example

As an illustration of the formulation presented in this paper and the variational setting outlined above we consider the following numerical simulation obtained through a Galerkin projection of the weak form (77). We use a finite element discretization in terms of bi-quadratic isoparametric finite elements with two-point uniformly reduced numerical integration, and refer to Simo and Vu-Quoc (1986b,c) for a detailed account of the numerical procedure.

We consider a right angle frame as shown in Fig. 2, whose deformation is symmetric with respect to the plane  $y-z$ . The apex of the frame is constrained to remain in the plane  $y-z$  at all time. The hinged ends can only slide on the  $x$ -axis, and rotate about the  $z$ -axis. Due to this symmetry, only half of the frame is modeled. The value of Young's modulus is  $E = 71240$ , and the value of the shear modulus is  $G = 27191$ . The cross-section has the following geometric properties:  $J_{11} = 1350$ ,  $J_{22} = 0.54$ ,  $\mathbb{J} = 2.16$  and  $\Xi = 40.5$ . A moment with increasing magnitude is applied at the hinged end of the frame. To trigger the out-of-plane bifurcation, a very small perturbation load is also applied at the apex. When the amplitude of the applied moment reaches its critical value, the frame buckles out-of-plane. At this moment, we remove the perturbation load, and employ a combination of displacement-control and arc-length method to trace the post-buckling response. The frame undergoes revolutions about the  $x$ -axis as we keep applying the end moments; two revolutions are performed in the present analysis. A perspective view of the first revolution is shown in Fig. 3, where the deformed shapes are given with no magnification, i.e. at the same scale as that of the geometry of the structure. A plot of the magnitude of the applied moment versus the lateral displacement of the apex is given in Fig. 4. It is fairly obvious that for this type of cross-section, the effect of warping is insignificant as compared with the formulation without warping in Simo and Vu-Quoc (1986c); Fig. 5 shows a zoomed-

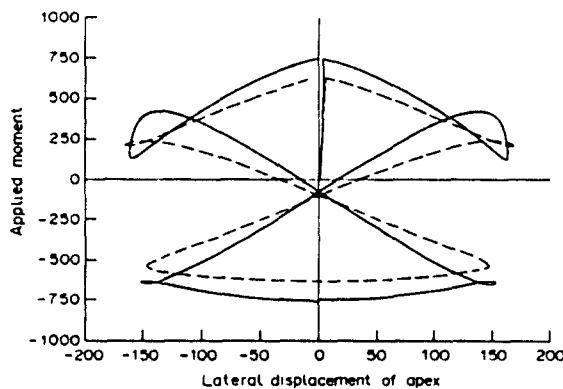


Fig. 6. Right angle frame. Applied moment versus lateral displacement of apex. Solution (*in solid line*) for warping inertia constant  $\Xi = 1000$ . The solution *without warping* is shown by the *dotted line*.

in portion of Fig. 4 with the result without warping in dotted line. Further, we note that the results remain essentially the same when warping at the hinged ends is constrained to zero or not. In Figure 6, the effect of warping is clearly demonstrated for a cross-section with a warping moment of inertia  $\Xi = 1000$ , but possessing the same other properties as the previous cross-section (the dotted line is the same plot as in Fig. 4 reproduced for comparison).

## 7. CLOSURE

We have presented a model of a finitely deformable beam accommodating shear and warping distortions of the cross-section. The geometry of deformations is described exactly through configurations that take values in  $\mathbb{R}^3 \times SO(3) \times \mathbb{R}$ ; a nonlinear differentiable manifold. The development of our model is based on basic concepts from three-dimensional elasticity: the mechanical power of the beam is exactly reduced from the stress power of the three-dimensional continuum, and provides an identification of the resultant forces and their conjugate strain measures. The model incorporates Vlasov's notions of bi-moment and bi-shear in a fully nonlinear geometrically exact context. In particular, the additional balance law for the bi-moment is exact (within the kinematic hypothesis) and not restricted to the usual case of second-order theories. Properly invariant reduced constitutive relations are motivated and developed within the framework of hyperelasticity. The variational formulation of the model is also discussed, including the linearization of the weak form and the structure of the local tangent operator, and is illustrated by means of a numerical example.

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## APPENDIX. PROOF OF IDENTITIES (16)

First, since  $\Delta f = f_{,\beta\beta} = 0$ , we have

$$f_{,x} = \delta_{x\beta} f_{,\beta} = X_{x,\beta} f_{,\beta} = [X_{x,\beta} f_{,\beta}]_{,\beta}. \quad (A1)$$

Use of Green's formula, boundary condition (13)<sub>2</sub>, and orthogonality conditions (14) yields

$$\int_{\Omega} f_{,x} \, d\Omega = -e_{\beta\gamma} \int_{\partial\Omega} X_{\gamma} (X_{\beta} - S_{\beta}) v_{,\gamma} \, d\Gamma = -e_{\beta\alpha} \int_{\Omega} (X_{\beta} - S_{\beta}) \, d\Omega = e_{\beta\alpha} A S_{\beta} \tag{A2}$$

which proves (15)<sub>1</sub>. Next, since  $e_{\alpha\beta} \delta_{\alpha\beta} = 0$ , use of Green's formula and boundary condition (13)<sub>2</sub> yields

$$e_{\alpha\beta} \int_{\Omega} (X_{\alpha} - S_{\alpha}) f_{,\beta} \, d\Omega = \int_{\Omega} [e_{\alpha\beta} (X_{\alpha} - S_{\alpha}) f]_{,\beta} \, d\Omega = - \int_{\partial\Omega} f f_{,\beta} v_{\beta} \, d\Gamma. \tag{A3}$$

Using again Green's formula along with the fact that  $f_{,\beta\beta} = 0$  and definition (15)<sub>2</sub> of the torsion modulus, from (A3) there follows

$$\int_{\Omega} [f_{,\beta}]^2 \, d\Omega = -e_{\alpha\beta} \int_{\Omega} (X_{\alpha} - S_{\alpha}) f'_{,\beta} \, d\Omega = J_S - \mathbb{J}. \tag{A4}$$

Identities (16)<sub>1,4</sub> easily follow from (A2), (A4) and definition (15)<sub>1</sub> of the torsion modulus. ■